NAG Toolbox for MATLAB

f08yl

1 Purpose

f08yl estimates condition numbers for specified eigenvalues and/or eigenvectors of a matrix pair in generalized real Schur form.

2 Syntax

```
[s, dif, m, info] = f08yl(job, howmny, select, a, b, vl, vr, mm, 'n', n)
```

3 Description

f08yl estimates condition numbers for specified eigenvalues and/or right eigenvectors of an n by n matrix pair (S, T) in real generalized Schur form. The function actually returns estimates of the reciprocals of the condition numbers in order to avoid possible overflow.

The pair (S, T) are in real generalized Schur form if S is block upper triangular with 1 by 1 and 2 by 2 diagonal blocks and T is upper triangular as returned, for example, by f08xa, or f08xe with **job** = 'S'. The diagonal elements, or blocks, define the generalized eigenvalues (α_i, β_i), for $i = 1, 2, \ldots, n$ of the pair (S, T) and the eigenvalues are given by

$$\lambda_i = \alpha_i / \beta_i,$$

so that

$$\beta_i S x_i = \alpha_i T x_i \qquad \text{or} \qquad S x_i = \lambda_i T x_i,$$

where x_i is the corresponding (right) eigenvector.

If S and T are the result of a generalized Schur factorization of a matrix pair (A, B)

$$A = QSZ^{\mathrm{T}}, \qquad B = QTZ^{\mathrm{T}}$$

then the eigenvalues and condition numbers of the pair (S, T) are the same as those of the pair (A, B).

Let $(\alpha, \beta) \neq (0, 0)$ be a simple generalized eigenvalue of (A, B). Then the reciprocal of the condition number of the eigenvalue $\lambda = \alpha/\beta$ is defined as

$$s(\lambda) = \frac{\left(\left| y^{\mathrm{T}} A x \right|^2 + \left| y^{\mathrm{T}} B x \right|^2 \right)^{1/2}}{\left(\|x\|_2 \|y\|_2 \right)},$$

where x and y are the right and left eigenvectors of (A, B) corresponding to λ. If both α and β are zero, then (A, B) is singular and $s(\lambda) = -1$ is returned.

The definition of the reciprocal of the estimated condition number of the right eigenvector x and the left eigenvector y corresponding to the simple eigenvalue λ depends upon whether λ is a real eigenvalue, or one of a complex conjugate pair.

If the eigenvalue λ is real and U and V are orthogonal transformations such that

$$U^{\mathrm{T}}(A, B)V = (S, T) = \begin{pmatrix} \alpha & * \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} \beta & * \\ 0 & T_{22} \end{pmatrix},$$

where S_{22} and T_{22} are $(n-1)$ by $(n-1)$ matrices, then the reciprocal condition number is given by

$$\mathrm{Dif}(x) \equiv \mathrm{Dif}(y) = \mathrm{Dif}((\alpha, \beta), (S_{22}, T_{22})) = \sigma_{\min}(Z),$$

where $\sigma_{\min}(Z)$ denotes the smallest singular value of the $2(n-1)$ by $2(n-1)$ matrix

$$Z = \begin{pmatrix} \alpha \otimes I & -1 \otimes S_{22} \\ \beta \otimes I & -1 \otimes T_{22} \end{pmatrix}$$

and $\otimes$ is the Kronecker product.

If λ is part of a complex conjugate pair and U and V are orthogonal transformations such that

$$U^{\mathrm{T}}(A, B)V = (S, T) = \begin{pmatrix} S_{11} & * \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} T_{11} & * \\ 0 & T_{22} \end{pmatrix},$$

where S_{11} and T_{11} are two by two matrices, S_{22} and T_{22} are $(n-2)$ by $(n-2)$ matrices, and (S_{11}, T_{11}) corresponds to the complex conjugate eigenvalue pair λ, $\bar{\lambda}$, then there exist unitary matrices U_1 and V_1 such that

$$U_1^H S_{11} V_1 = \begin{pmatrix} s_{11} & s_{12} \\ 0 & s_{22} \end{pmatrix} \quad \text{and} \quad U_1^H T_{11} V_1 = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}.$$

The eigenvalues are given by $\lambda = s_{11}/t_{11}$ and $\bar{\lambda} = s_{22}/t_{22}$. Then the Frobenius norm-based, estimated reciprocal condition number is bounded by

$$\mathrm{Dif}(x) \equiv \mathrm{Dif}(y) \leq \min(d_1, \max(1, |\mathrm{Re}(s_{11})/\mathrm{Re}(s_{22})|), d_2)$$

where $\mathrm{Re}(z)$ denotes the real part of z, $d_1 = \mathrm{Dif}((s_{11}, t_{11}), (s_{22}, t_{22})) = \sigma_{\min}(Z_1)$, Z_1 is the complex two by two matrix

$$Z_1 = \begin{pmatrix} s_{11} & -s_{22} \\ t_{11} & -t_{22} \end{pmatrix},$$

and d_2 is an upper bound on $\mathrm{Dif}((S_{11}, T_{11}), (S_{22}, T_{22}))$; that is, an upper bound on $\sigma_{\min}(Z_2)$, where Z_2 is the $(2n-2)$ by $(2n-2)$ matrix

$$Z_2 = \begin{pmatrix} S_{11}^T \otimes I & -I \otimes S_{22} \\ T_{11}^T \otimes I & -I \otimes T_{22} \end{pmatrix}.$$

See Sections 2.4.8 and 4.11 of Anderson *et al.* 1999 and Kagström and Poromaa 1996 for further details and information.

4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D 1999 *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia URL: http://www.netlib.org/lapack/lug

Kagström B and Poromaa P 1996 LAPACK-style algorithms and software for solving the generalized Sylvester equation and estimating the separation between regular matrix pairs *ACM Trans. Math. Software* **22** 78–103

5 Parameters

5.1 Compulsory Input Parameters

1: **job – string**

Indicates whether condition numbers are required for eigenvalues and/or eigenvectors.

job = 'E'

Condition numbers for eigenvalues only are computed.

job = 'V'

Condition numbers for eigenvectors only are computed.

job = 'B'

Condition numbers for both eigenvalues and eigenvectors are computed.

Constraint: **job** = 'E', 'V' or 'B'.

2: **howmny – string**

Indicates how many condition numbers are to be computed.

howmny = 'A'

Condition numbers for all eigenpairs are computed.

howmny = 'S'

Condition numbers for selected eigenpairs (as specified by **select**) are computed.

Constraint: **howmny** = 'A' or 'S'.

3: **select**$(*)$ **– logical array**

Note: the dimension of the array **select** must be at least $\max(1, \mathbf{n})$ if **howmny** = 'S', and at least 1 otherwise.

Specifies the eigenpairs for which condition numbers are to be computed if **howmny** = 'S'. To select condition numbers for the eigenpair corresponding to the real eigenvalue λ_j, **select**(j) must be set **true**. To select condition numbers corresponding to a complex conjugate pair of eigenvalues λ_j and λ_{j+1}, **select**(j) and/or **select**$(j+1)$ must be set to **true**.

If **howmny** = 'A', **select** is not referenced.

4: **a**(**lda**,$*$) **– double array**

The first dimension of the array **a** must be at least $\max(1, \mathbf{n})$

The second dimension of the array must be at least $\max(1, \mathbf{n})$

The upper quasi-triangular matrix S.

5: **b**(**ldb**,$*$) **– double array**

The first dimension of the array **b** must be at least $\max(1, \mathbf{n})$

The second dimension of the array must be at least $\max(1, \mathbf{n})$

The upper triangular matrix T.

6: **vl**(**ldvl**,$*$) **– double array**

The first dimension, **ldvl**, of the array **vl** must satisfy

if **job** = 'E' or 'B', **ldvl** $\geq \max(1, \mathbf{n})$;
ldvl ≥ 1 otherwise.

The second dimension of the array must be at least $\max(1, \mathbf{mm})$ if **job** = 'E' or 'B', and at least 1 otherwise

If **job** = 'E' or 'B', **vl** must contain left eigenvectors of (S, T), corresponding to the eigenpairs specified by **howmny** and **select**. The eigenvectors must be stored in consecutive columns of **vl**, as returned by f08yk or f08wa.

If **job** = 'V', **vl** is not referenced.

7: **vr(ldvr,$*$) – double array**

The first dimension, **ldvr**, of the array **vr** must satisfy

if **job** = 'E' or 'B', **ldvr** $\geq \max(1, \mathbf{n})$;
ldvr ≥ 1 otherwise.

The second dimension of the array must be at least $\max(1, \mathbf{mm})$ if **job** = 'E' or 'B', and at least 1 otherwise

If **job** = 'E' or 'B', **vr** must contain right eigenvectors of (S, T), corresponding to the eigenpairs specified by **howmny** and **select**. The eigenvectors must be stored in consecutive columns of **vr**, as returned by f08yk or f08wa.

If **job** = 'V', **vr** is not referenced.

8: **mm – int32 scalar**

The number of elements in the arrays **s** and **dif**.

Constraint: **mm** $\geq$ **m**.

5.2 Optional Input Parameters

1: **n – int32 scalar**

Default: The first dimension of the arrays **a**, **b** and the second dimension of the arrays **a**, **b**. (An error is raised if these dimensions are not equal.)

n, the order of the matrix pair (S, T).

Constraint: **n** ≥ 0.

5.3 Input Parameters Omitted from the MATLAB Interface

lda, ldb, ldvl, ldvr, work, lwork, iwork

5.4 Output Parameters

1: **s($*$) – double array**

Note: the dimension of the array **s** must be at least $\max(1, \mathbf{mm})$.

If **job** = 'E' or 'B', the reciprocal condition numbers of the selected eigenvalues, stored in consecutive elements of the array. For a complex conjugate pair of eigenvalues two consecutive elements of **s** are set to the same value. Thus **s**(j), **dif**(j), and the jth columns of VL and VR all correspond to the same eigenpair (but not in general the jth eigenpair, unless all eigenpairs are selected).

If **job** = 'V', **s** is not referenced.

2: **dif($*$) – double array**

Note: the dimension of the array **dif** must be at least $\max(1, \mathbf{mm})$.

If **job** = 'V' or 'B', the estimated reciprocal condition numbers of the selected eigenvectors, stored in consecutive elements of the array. For a complex eigenvector two consecutive elements of **dif** are set to the same value. If the eigenvalues cannot be reordered to compute **dif**(j), **dif**(j) is set to 0; this can only occur when the true value would be very small anyway.

If **job** = 'E', **dif** is not referenced.

3: **m – int32 scalar**

The number of elements of the arrays **s** and **dif** used to store the specified condition numbers; for each selected real eigenvalue one element is used, and for each selected complex conjugate pair of eigenvalues, two elements are used. If **howmny** = 'A', **m** is set to **n**.

4: **info – int32 scalar**

info $= 0$ unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

info $= -i$

If **info** $= -i$, parameter i had an illegal value on entry. The parameters are numbered as follows:

1: **job**, 2: **howmny**, 3: **select**, 4: **n**, 5: **a**, 6: **lda**, 7: **b**, 8: **ldb**, 9: **vl**, 10: **ldvl**, 11: **vr**, 12: **ldvr**, 13: **s**, 14: **dif**, 15: **mm**, 16: **m**, 17: **work**, 18: **lwork**, 19: **iwork**, 20: **info**.

It is possible that **info** refers to a parameter that is omitted from the MATLAB interface. This usually indicates that an error in one of the other input parameters has caused an incorrect value to be inferred.

7 Accuracy

None.

8 Further Comments

An approximate asymptotic error bound on the chordal distance between the computed eigenvalue $\tilde{\lambda}$ and the corresponding exact eigenvalue λ is

$$\chi\left(\tilde{\lambda}, \lambda\right) \leq \epsilon \|(A, B)\|_F / S(\lambda)$$

where ϵ is the ***machine precision***.

An approximate asymptotic error bound for the right or left computed eigenvectors $\tilde{x}$ or $\tilde{y}$ corresponding to the right and left eigenvectors x and y is given by

$$\theta(\tilde{z}, z) \leq \epsilon \|(A, B)\|_F / \mathrm{Dif}.$$

The complex analogue of this function is f08yy.

9 Example

```
job = 'Both';
howmny = 'All';
select = [false];
a = [4, 1, 1, 2;
     0, 3, -1, 1;
     0, 1, 3, 1;
     0, 0, 0, 6];
b = [2, 1, 1, 3;
     0, 1, 0, 1;
     0, 0, 1, 1;
     0, 0, 0, 2];
vl = [0.4, 0, 0, 0;
     0, -0, 0.5, 0;
     0.4, 0.5, 0, 0;
     1, 0.25, -0.75, 1];
vr = [1, -0.6666666666666666, -0.3333333333333333, -1;
     0, -0, 0.6666666666666666, 0.5714285714285718;
     0, 0.6666666666666666, 0, -0.5714285714285724;
     0, 0, 0, 0.285714285714286];
mm = int32(4);
[s, dif, m, info] = f08yl(job, howmny, select, a, b, vl, vr, mm)
```

```
s =
    1.5570
    1.7345
    1.7345
    1.3720
dif =
    0.5435
    0.1514
    0.1514
    0.1245
m =
           4
info =
           0
```